PROOF — *Answers*

1 **a** e.g. $x = \frac{1}{8} \Rightarrow \sqrt[3]{x} = \frac{1}{2}, \quad \frac{1}{2} > \frac{1}{8}$

[any value of x in the interval $0 < x < 1$]

b e.g. $n = 7 \Rightarrow n^3 - n + 7 = 7(49 - 1 + 1)$ which is divisible by 7

[many other examples]

2 assume $\sqrt{\pi}$ is rational $\Rightarrow \sqrt{\pi} = \frac{p}{q}, \quad p, q \in \mathbb{Z}$

$\Rightarrow \pi = \frac{p^2}{q^2}, \quad p^2, q^2 \in \mathbb{Z} \quad \therefore \pi$ rational

$\Rightarrow$ contradiction $\therefore \sqrt{\pi}$ irrational

3 consider $15x^2 - 11x + 2 < 0$

$\Rightarrow (5x - 2)(3x - 1) < 0$

$\Rightarrow \frac{1}{3} < x < \frac{2}{5}$

e.g. $x = 0.35 \Rightarrow 15x^2 - 11x + 2 = -0.0125, \quad -0.0125 < 0$

[any value of x in the interval $\frac{1}{3} < x < \frac{2}{5}$]

4 **a** $n^2 + 2n = (2m + 1)^2 + 2(2m + 1)$

$= 4m^2 + 4m + 1 + 4m + 2$

$= 4m^2 + 8m + 3$

b assume $n^2 + 2n$ even and n odd, where $n \in \mathbb{Z}$

n odd $\Rightarrow n = 2m + 1, \quad m \in \mathbb{Z}$

$\Rightarrow n^2 + 2n = 4m^2 + 8m + 3 = 2(2m^2 + 4m + 1) + 1$

$2m^2 + 4m + 1 \in \mathbb{Z} \quad \therefore n^2 + 2n$ odd

$\Rightarrow$ contradiction $\therefore n$ even

5 **a** $k\cos x - \operatorname{cosec} x = 0 \Rightarrow k\cos x = \frac{1}{\sin x}$

$\Rightarrow k\sin x\cos x = 1$

$\Rightarrow \frac{1}{2}k\sin 2x = 1$

$\Rightarrow \sin 2x = \frac{2}{k}$

$|\sin 2x| \le 1 \Rightarrow \left|\frac{2}{k}\right| \le 1$

$\Rightarrow |k| \ge 2$

b $3\cos x - \operatorname{cosec} x = 0 \Rightarrow \sin 2x = \frac{2}{3}$

$2x = 41.810,\ 180 - 41.810,\ 360 + 41.810,\ 540 - 41.810$

$2x = 41.810,\ 138.190,\ 401.810,\ 498.190$

$x = 20.9,\ 69.1,\ 200.9,\ 249.1$ (1dp)

6 assume $x^2 - y^2 = 1$, where $x, y \in \mathbb{Z}^+$

$x^2 - y^2 = 1 \Rightarrow (x + y)(x - y) = 1$

$x, y \in \mathbb{Z}^+ \Rightarrow (x + y), (x - y) \in \mathbb{Z}$ and $(x + y) > 0$

$\therefore\ x + y = 1$ and $x - y = 1$

adding $\Rightarrow 2x = 2$

$\Rightarrow x = 1$

$\Rightarrow y = 0$

$\Rightarrow$ contradiction $\therefore$ no positive integer solutions

7 **a** false e.g. $a = \sqrt{2}$, $b = 1 - \sqrt{2}$ $\Rightarrow$ a and b irrational

and $a + b = 1$ which is rational

[many other examples]

b true m, n consecutive odd integers $\Rightarrow m = 2a + 1, n = 2a + 3,\ a \in \mathbb{Z}$

$\Rightarrow m + n = 2a + 1 + 2a + 3 = 4a + 4 = 4(a + 1)$

$a + 1 \in \mathbb{Z}$ $\therefore$ $m + n$ divisible by 4

c false e.g. $x = \frac{5\pi}{3} \Rightarrow \cos x = \frac{1}{2}$ and $1 + \sin x = 1 - \frac{\sqrt{3}}{2}$, $\frac{1}{2} > 1 - \frac{\sqrt{3}}{2}$

[any value of x of the form $2n\pi + y$, $n \in \mathbb{Z}$, $-\frac{\pi}{2} < y < 0$]

8 **a** $\log_2 3 = \frac{p}{q} \Rightarrow 2^{\frac{p}{q}} = 3$

$\Rightarrow (2^{\frac{p}{q}})^q = 3^q$

$\Rightarrow 2^p = 3^q$

b assume $\log_2 3$ is rational $\Rightarrow \log_2 3 = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

$\Rightarrow 2^p = 3^q$

2 and 3 are co-prime $\Rightarrow p = q = 0$

$\Rightarrow$ contradiction $\therefore \log_2 3$ is irrational

c e.g. $a = 2, b = \sqrt{2} \Rightarrow a$ rational and b irrational

and $\log_a b = \frac{1}{2}$ which is rational

[many other examples]

9 **a** $y = \frac{x-2}{4x}$ swap $x = \frac{y-2}{4y}$

$4xy = y - 2$

$y(4x - 1) = -2$

$y = \frac{2}{1-4x}$

$f^{-1}(x) = \frac{2}{1-4x}$, $x \in \mathbb{R}$, $x \neq \frac{1}{4}$

b $f(x) = f^{-1}(x) \Rightarrow \frac{x-2}{4x} = \frac{2}{1-4x}$

$\Rightarrow (x - 2)(1 - 4x) = 8x$

$\Rightarrow 4x^2 - x + 2 = 0$

$b^2 - 4ac = 1 - 32 = -31$

$b^2 - 4ac < 0 \Rightarrow$ no real roots

$\therefore$ no real values of x for which $f(x) = f^{-1}(x)$